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An isothermal theory of anisotropic rods

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SUMMARY

To provide an isothermal, deterministic theory of anisotropic rods is the primary objective of this paper. Our starting point is the 3-D linear theory of micropolar elastodynamics. First, the governing equations of the theory are established by the use of a suitable averaging procedure together with a separation of variables solution for kinematic variables. Next, without making the usual definiteness assumption for the strain energy density, a dynamic uniqueness theorem is constructed for the solutions of the governing equations. Logarithmic convexity arguments are then used to enumerate a set of conditions sufficient for uniqueness. The theory includes the effects of warping and shearing deformations, and in fact, it incorporates as many higher order effects as deemed necessary in any special case. Also, the application of the theory is illustrated in a sample example.

Notation

Throughout the text, we use standard Cartesian tensor notation in an Euclidean 3-space $\mathscr{E}$. The micropolar rod is embedded in this space. Einstein's summation convention is implied for all repeated Latin indices (1, 2, 3) and Greek indices (2, 3) unless indices are enelosed with parentheses. A comma followed by an index stands for partial differentiation with respect to the indicated coordinate x_k. A superposed dot denotes time differentiation. A single prime is used to designate partial differentiation with respect to the axial coordinate $x_1 \equiv z$. A star indicates prescribed quantities. Further, the Cartesian product of a region $\mathscr{B}$ and the time interval $[t_0, T)$ is denoted by $\mathscr{B} \times [t_0, T)$, where $T > t_0$ may be infinity. The symbol $\mathscr{B}(t)$ refers to the region $\mathscr{B}$ at time t.

Nomenclature

$\mathscr{E}$	Euclidean 3-space
x_k, $x_1 \equiv z$, x_α	a system of right-handed Cartesian coordinates in $\mathscr{E}$, rod axis, lateral coordinates; $k = 1, \alpha$ $(\alpha = 2, 3)$
$\mathscr{B}, \overline{\mathscr{B}}, \partial\mathscr{B}$	a regular region of space in $\mathscr{E}$, its closure and boundary surface
$\mathscr{S}_d, \mathscr{S}_\sigma$	complementary subsets of $\partial\mathscr{B}$, on which deformations and stresses are prescribed, respectively; $\mathscr{S}_d \cup \mathscr{S}_\sigma = \mathscr{S}$, $\mathscr{S}_d \cap \mathscr{S}_\sigma = 0$
L	length of rod
$\mathscr{A}$	area of cross-section of rod
$\mathscr{C}$	a Jordan curve which bounds $\mathscr{A}$
$\mathscr{V}, \mathscr{S}$	entire volume of rod and its boundary surface
$\mathscr{S}_l, \mathscr{S}_t$	lateral surface of rod, surface portion of $\mathscr{S}_l$ on which stresses are prescribed
$\mathscr{A}_r, \mathscr{A}_l$	right and left faces of rod
dv, dA, ds	element of volume and area on $\mathscr{A}$, and line element along $\mathscr{C}$
n_i, ν_i	unit exterior normal vectors to $\mathscr{S}$ and $\mathscr{C}$
t	time
t_{kl}, m_{kl}	components of stress and couple stress tensors
f_i, l_i	body force and body couple vectors per unit volume
ρ, J_{kl}	mass density, components of microinertia tensor
u_i, φ_i	displacement and microrotation vectors

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Θ	prescribed steady temperature increment
δ_{kl}, ε_{klm}	components of Kronecker's delta and alternating tensor
t_i, m_i	stress and couple stress vectors
ε_{kl}, e_{kl}	components of strain and infinitesimal strain tensors
C_{klmn}, D_{klmn}	components of isothermal elastic stiffnesses
B_{kl}	thermal coefficients of material
λ, μ	Lamé's elasticity constants
α, β, γ, κ	elastic moduli of microisotropic continuum
B	coefficient of linear thermal expansion
d_i, d_{kl}	u_i or φ_i, ε_{kl} or e_{kl} and/or γ_{kl}
$T_{kl}^{(m,n)}$, $M_{kl}^{(m,n)}$	components of stress and couple stress resultants of order (m, n)
$T_k^{(m,n)}$ $M_k^{(m,n)}$	components of stress and couple stress vector resultants of order (m, n)
$d_k^{(m,n)}$, $d_{kl}^{(m,n)}$	components of deformation ($u_k^{(m,n)}$, $\varphi_k^{(m,n)}$) and strain ($\varepsilon_{kl}^{(m,n)}$, $e_{kl}^{(m,n)}$, $\gamma_{kl}^{(m,n)}$) of order (m, n)
$F_i^{(m,n)}$, $L_i^{(m,n)}$	body force and body couple resultants of order (m, n)
$U_i^{(m,n)}$, $\phi_i^{(m,n)}$	displacement and microrotation resultants of order (m, n)
$P_i^{(m,n)}$, $Q_i^{(m,n)}$	effective loads of order (m, n)
$\mathcal{N}$, $\mathcal{M}$, P, Q, w, ϕ	$T_{11}^{(0,0)}$, $M_{11}^{(0,0)}$, $P_1^{(0,0)}$, $Q_1^{(0,0)}$, $U_1^{(0,0)}$, $\varphi_1^{(0,0)}$
v_0	rod velocity, $(E/\rho)^{\frac{1}{2}}$
K, W	kinetic and potential energy densities
V, U	kinetic and potential energies per unit length of rod
Ω	total energy of rod
$C^{(m,n)}$	functions with derivatives of order up to and including (m) and (n) with respect to space coordinates and time, respectively
G(t)	logarithmic convexity function
$(\ )^{\cdot}$	time differentiation, $\partial/\partial t(\ )$
$(\ )'$	partial differentiation with respect to the axial coordinate, $\partial/\partial z(\ )$
E, ν	Young's modulus, Poisson's ratio

1. Introduction

In the literature of recent years, considerable attention has been focused on the formulation of the one- and two-dimensional continuum theories by the reduction of the 3-D elastodynamics. In regard to the literature on this study, though by no means exhaustive, we mention, in particular, two recent works: an excellent article on the present status of rods [1] and a general theory of elastic non-polar beams [2]. The latter work is now supplemented and amplified to govern all the types of motion of anisotropic rods on the basis of the 3-D linear theory of micropolar elastodynamics.

In what follows, we consider a slender micropolar rod of uniform cross-section. By a micropolar rod we simply mean a rod made of certain class of materials with microstructure. Due to its granular and fibrous structure, this type of materials can support couple or moment stresses, and its deformations at each point consist of both displacements and microrotations (see, e.g., [3] and references therein). Throughout the micropolar rod space, we take for granted that all stresses and deformations are continuous. Further, we assume that all the field quantities do not vary widely over the cross-section of the rod. Owing to these considerations, we use a separation of variables solution for kinematical variables and a suitable averaging procedure, with automatic rationality of the resultant one-dimensional rod theory. The theory gives rise to new types of waves not encountered in any of the classical rod theories. In particular, the nature of extensional waves is discussed in a simple example.

Briefly stated, a résumé of the basic equations of linear micropolar elasticity theory is presented in the next section. Section 3 deals with the geometry and deformations of rod. The one-dimensional, higher order theory is systematically established in Section 4. Bernoulli's theory of micropolar rods is obtained as a special case of the isotropic theory in Section 5.

A sample example is then carried out in detail to illustrate simply the application of this approximate theory. In Section 6, the initial and boundary conditions sufficient for a unique solution of the governing equations of the isothermal theory are examined by the use of logarithmic convexity arguments. The results are briefly discussed in the last section.

2. Linear fundamental equations

In this paper all the field equations are evidently supposed to be small so that the linear governing equations of micropolar elasticity theory can be applied. We then summarize, as already established, the following basic equations of the linear theory. A complete account of these equations is given, for instance, in [3, 4].

When the motion of micropolar continuum $\mathscr{B}+\partial\mathscr{B}$ with boundary $\partial\mathscr{B}$ ($\partial\mathscr{B}=\mathscr{S}_d \cup \mathscr{S}_\sigma$, $\mathscr{S}_d \cap \mathscr{S}_\sigma=0$) is referred to a x_k-fixed system of rectangular Cartesian coordinates in the Euclidean 3-space $\mathscr{E}$, we may express the local balance of momenta:

$$\left.\begin{aligned} & t_{kl,k}+f_l-\rho\ddot{u}_l=0 \\ & m_{kl,k}+\varepsilon_{lkm}t_{km}+l_l-\rho J_{kl}\ddot{\varphi}_k=0 \end{aligned}\right\} \text{on } \mathscr{B}\times[t_0, T) \tag{2.1}$$

the constitutive equations for an anisotropic body:

$$\left.\begin{aligned} & t_{kl}=C_{klmn}\varepsilon_{mn}+B_{kl}\Theta \\ & m_{kl}=D_{lkmn}\gamma_{mn} \end{aligned}\right\} \text{ on } \mathscr{B}\times[t_0, T) \tag{2.2}$$

with

$$C_{klmn}=C_{mnkl},\ D_{klmn}=D_{mnkl} \tag{2.3}$$

the strain-deformation relations:

$$\left.\begin{aligned} & \gamma_{mn}=\varphi_{m,n} \\ & \varepsilon_{kl}=u_{l,k}+\varepsilon_{lkm}\varphi_m \end{aligned}\right\} \text{on } \mathscr{B}\times[t_0, T) \tag{2.4}$$

the boundary conditions:

$$u_k-u_k^*=0\,,\ \varphi_k-\varphi_k^*=0 \text{ on } \mathscr{S}_d\times[t_0, T) \tag{2.5}$$

and

$$t_k-t_k^*=0\,,\ m_k-m_k^*=0 \text{ on } \mathscr{S}_\sigma\times[t_0, T) \tag{2.6}$$

with

$$t_l=n_k t_{kl}\,,\ m_l=n_k m_{kl} \tag{2.7}$$

and a Cauchy data of the form

$$\left.\begin{aligned} & u_k(x_i, t_0)-\alpha_k^*(x_i)=0\,,\ \dot{u}_k(x_i, t_0)-\beta_k^*(x_i)=0 \\ & \varphi_k(x_i, t_0)-\gamma_k^*(x_i)=0\,,\ \dot{\varphi}_k(x_i, t_0)-\delta_k^*(x_i)=0 \end{aligned}\right\} \text{on } \mathscr{B}(t_0) \tag{2.8}$$

for the initial conditions. In the above equations, t_{kl}, m_{kl}, t_i and m_i, in this order, denote the components of the stress and couple stress tensors, and the stress and couple stress vectors. ρ is the mass density, u_i the displacement vector, f_i the body force vector, ε_{klm} the components of the alternating tensor, φ_i the microrotation vector, l_i the body couple vector, J_{kl} the components of the microinertia tensor. C_{klmn} and D_{klmn} stand for the components of the isothermal elastic stiffnesses, ε_{kl} and γ_{kl} for the components of the strain tensor, Θ for a prescribed steady temperature increment, and B_{kl} for the thermal coefficients of material. $\mathscr{S}_d$ and $\mathscr{S}_\sigma$ are the complementary regular subsurfaces of $\partial\mathscr{B}$, where the deformations and the stresses are prescribed, respectively, and n_i is the unit exterior vector normal to $\partial\mathscr{B}$. Further, u_i^*, α_i^* and β_i^*, φ_i^*, γ_i^* and δ_i^*, t_i^* and m_i^* are used to designate the prescribed vector functions.

Moreover, let $u_i\in C^{(1,2)}$, $\varphi_i\in C^{(1,2)}$, $f_i\in C^{(0,0)}$, $l_i\in C^{(0,0)}$, $t_{kl}\in C^{(1,0)}$, $m_{kl}\in C^{(1,0)}$, $\varepsilon_{kl}\in C^{(0,0)}$ and $\gamma_{kl}\in C^{(0,0)}$. Here, $C^{(m,n)}$ represents the functions with derivatives of order up to and including (m) and (n) with respect to the space coordinates and time, respectively, provided that the functions and their derivatives exist and are continuous on $\overline{\mathscr{B}}\times[t_0, T)$. $\overline{\mathscr{B}}$ indicates the closure of the regular region of space $\mathscr{B}\in\mathscr{E}$ at $t=t_0$.

We also note that the constitutive coefficients in Equations (2.2), when specialized to isotropy, are of the form:

$$\left.\begin{aligned} C_{klmn} &= \lambda\delta_{kl}\delta_{mn}+(\mu+\kappa)\delta_{km}\delta_{ln}+\mu\delta_{kn}\delta_{lm} \\ D_{klmn} &= \alpha\delta_{kl}\delta_{mn}+\beta\delta_{kn}\delta_{lm}+\gamma\delta_{km}\delta_{ln} \\ B_{kl} &= B\delta_{kl} \end{aligned}\right\} \tag{2.9}$$

where λ and μ denote Lamé's elasticity constants, B the coefficient of linear thermal expansion, and α, β, γ and κ are the four additional elastic moduli for micropolar continuum. In view of Equations (2.2) and (2.9) the isotropic constitutive equations may be expressed as

$$\left.\begin{aligned} t_{kl} &= \lambda e_{rr}\delta_{kl}+(\mu+\kappa)\varepsilon_{kl}+\mu\varepsilon_{ik}-B\Theta\delta_{kl} \\ m_{kl} &= \alpha\gamma_{rr}\delta_{kl}+\beta\gamma_{kl}+\gamma\gamma_{lk} \end{aligned}\right\} \text{ on } \mathscr{B}\times[t_0, T) \tag{2.10}$$

with the linear strain tensor:

$$e_{kl}=\tfrac{1}{2}(u_{k,l}+u_{l,k}) \text{ on } \mathscr{B}\times[t_0, T) \tag{2.11}$$

In addition, we recall that the microinertia tensor becomes $J_{kl}=J\delta_{kl}$ where J being a constant, for the microisotropic solid.

3. Kinematics

With reference to the x_k-system of Cartesian coordinates in the Euclidean 3-space $\mathscr{E}$, we consider a thin cylindrical rod $\mathscr{V}+\mathscr{S}$ with its smooth boundary surface $\mathscr{S}$, bounded by the right and left plane faces, $\mathscr{A}_r(x_1=L)$ and $\mathscr{A}_l(x_1=0)$, and the lateral surface $\mathscr{S}_l(f(x_\alpha)=0)$. The x_k-axes are located at the centroid of the initial cross-section of rod; the x_1-axis is chosen to be the centroidal rod axis, while the x_α-axes indicate the principal axes of cross-sections.

The deformation components d_k (u_k or φ_k) of a generic point $P(x_i)\in\mathscr{V}$ under the usual assumptions of rods, can consistently be represented as

$$d_k(x_1, x_\alpha, t) = \sum_{m=0}^{M=\infty}\sum_{n=0}^{N=\infty} P_m^{(k)}(x_2)\,Q_n^{(k)}(x_3)\,d_k^{(m,n)}(x_1, t) \text{ on } \mathscr{V}\times[t_0, T) \tag{3.1a}$$

with

$$P_m^{(k)}(x_2)=x_2^m,\quad Q_n^{(k)}(x_3)=x_3^n \tag{3.1b}$$

in which the vector functions $d_i^{(m,n)}\in C^{(1,2)}$ are taken to exist, and they are as yet undetermined and independent functions in the rod length $\mathscr{L}$. Equations (3.1) clearly treat the rod as being a one-dimensional continuum. Further, the deformation field is sufficiently general to abrogate the usual Bernoulli–Euler hypotheses of rods (see, e.g., [5,6]), and hence there is no need to include the customary correction factors [5,7].

In view of Equations (2.4) and the series expansions (3.1), we obtain the strain distribution of the form:

$$d_{kl} = \sum_{m=0}^{M}\sum_{n=0}^{N} x_2^m x_3^n d_{kl}^{(m,n)}(x_1, t) \tag{3.2}$$

where

$$\begin{aligned} 2e_{kl}^{(m,n)} &= u_k'^{(m,n)}\delta_{1l}+u_l'^{(m,n)}\delta_{1k}+(m+1)(u_k^{(m+1,n)}\delta_{2l}+u_l^{(m+1,n)}\delta_{2k}) \\ &\quad+(n+1)(u_k^{(m,n+1)}\delta_{3l}+u_l^{(m,n+1)}\delta_{3k}) \end{aligned}$$

$$\left.\begin{aligned} \varepsilon_{kl}^{(m,n)} &= \varepsilon_{lkp}\varphi_p^{(m,n)}+u_l'^{(m,n)}\delta_{1k}+(m+1)u_l^{(m+1,n)}\delta_{2k} \\ &\quad+(n+1)u_l^{(m,n+1)}\delta_{3k} \\ \gamma_{kl}^{(m,n)} &= \varphi_k'^{(m,n)}\delta_{1l}+(m+1)\varphi_k^{(m+1,n)}\delta_{2l}+(n+1)\varphi_k^{(m,n+1)}\delta_{3l} \end{aligned}\right\} \text{ on } \mathscr{L}\times[t_0, T) \tag{3.3}$$

In Equation (3.2), d_{kl} is used to denote e_{kl}, ε_{kl} and γ_{kl}.

4. Rod equations

In this section, using the deformation field (3.1) together with a suitable averaging procedure [8,9] as a starting point, we construct the complete set of the micropolar rod equations from the 3-D equations of Section 2.

4.1. *Rod equations of motion*

To begin with we multiply the local equations of motion (2.1) by $x_2^m x_3^n$ and integrate over the cross-section of the rod, $\mathscr{A}$. After converting some of the surface integrals over $\mathscr{A}$ to the line integrals around the contour $\mathscr{C}$ by applying Green's transformation theorems, we then obtain the rod equations of motion:

$$\left.\begin{aligned} &T_{1k}^{\prime(m,n)} - mT_{2k}^{(m-1,n)} - nT_{3k}^{(m,n-1)} + P_k^{(m,n)} - \rho\ddot{U}_k^{(m,n)} = 0 \\ &M_{1k}^{\prime(m,n)} - mM_{2k}^{(m-1,n)} - nM_{3k}^{(m,n-1)} + \varepsilon_{klp}T_{lp}^{(m,n)} \\ &\qquad + Q_k^{(m,n)} - \rho J_{lk}\ddot{\phi}_l^{(m,n)} = 0 \end{aligned}\right\} \text{ on } \mathscr{L}\times[t_0, T) \tag{4.1}$$

Here, we have defined the stress and couple stress resultants of order (m, n):

$$(T_{kl}^{(m,n)}, M_{kl}^{(m,n)}) = \int_{\mathscr{A}} x_2^m x_3^n (t_{kl}, m_{kl})\,dA \tag{4.2}$$

the aerial moment of inertia of order (m, n):

$$A_{mn} = \int_{\mathscr{A}} x_2^m x_3^n\,dA \tag{4.3}$$

the acceleration resultants of order (m, n):

$$(\ddot{U}_k^{(m,n)}, \ddot{\phi}_k^{(m,n)}) = \sum_{p=0}^{M}\sum_{q=0}^{N} A_{m+p,n+q}(\ddot{u}_k^{(p,q)}, \ddot{\varphi}_k^{(p,q)}) \tag{4.4}$$

the body force and body couple resultants of order (m, n):

$$(F_i^{(m,n)}, L_i^{(m,n)}) = \int_{\mathscr{A}} x_2^m x_3^n (f_i, l_i)\,dA \tag{4.5}$$

the boundary forcing terms of order (m, n), arising from the moments of applied tractions and surface couples on $\mathscr{S}_1$ as

$$(S_i^{(m,n)}, R_i^{(m,n)}) = \oint_{\mathscr{C}} x_2^m x_3^n (t_i, m_i)\,ds \tag{4.6}$$

and finally the effective loads of order (m, n):

$$P_i^{(m,n)} = F_i^{(m,n)} + S_i^{(m,n)}, \quad Q_i^{(m,n)} = L_i^{(m,n)} + R_i^{(m,n)}. \tag{4.7}$$

In the most general case, the quantities defined in Equations (4.2–7) may be functions of x_1 and t.

4.2. *Constitutive relations*

With the aid of Equations (2.2) and (3.3), we find that the constitutive relations for the aforementioned stress and couple stress resultants are of the form:

$$\left.\begin{aligned} &T_{kl}^{(m,n)} = \sum_{p=0}^{M}\sum_{q=0}^{N} (C_{klrs}^{(m+p,n+q)}\varepsilon_{rs}^{(p,q)} + B_{kl}^{(m+p,n+q)}\Theta^{(p,q)}) \\ &M_{kl}^{(m,n)} = \sum_{p=0}^{M}\sum_{q=0}^{N} D_{klrs}^{(m+p,n+q)}\gamma_{rs}^{(p,q)} \end{aligned}\right\} \text{ on } \mathscr{L}\times[t_0, T) \tag{4.8}$$

Here, we have introduced the heterogeneous rod stiffnesses:

$$\left(C_{klrs}^{(m,n)}, D_{klrs}^{(m,n)}, B_{kl}^{(m,n)}\right) = \int_{\mathscr{A}} x_2^m x_3^n (C_{klrs}, D_{lkrs}, B_{kl})\,dA \tag{4.9}$$

and the temperature increment in the form:

$$\Theta(x_i) = \sum_{m=0}^{M} \sum_{n=0}^{N} x_2^m x_3^n \Theta^{(m,n)}(x_1) \tag{4.10}$$

In the case of isotropic micropolar material, the generalized linear constitutive relations (4.8) reduce to

$$\left.\begin{aligned} T_{kl}^{(m,n)} &= \sum_{p=0}^{M} \sum_{q=0}^{N} \qquad \delta_{kl} + (\mu+\kappa)\varepsilon_{kl}^{(p,q)} + \mu\varepsilon_{lk}^{(p,q)} - B\Theta^{(p,q)}\delta_{kl}]A_{m+p,n+q} \\ M_{kl}^{(m,n)} &= \sum_{p=0}^{M} \sum_{q=0}^{N} (\alpha\gamma_{rr}^{(p,q)}\delta_{kl} + \beta\gamma_{kl}^{(p,q)} + \gamma\gamma_{lk}^{(p,q)})A_{m+p,n+q} \end{aligned}\right\} \text{ on } \mathscr{L} \times [t_0, T) \tag{4.11}$$

in which Equations (2.9) are employed.

4.3. *Boundary conditions*

Let $\mathscr{S}_d$ stand for some portion of the lateral surface $\mathscr{S}_1$, and $\mathscr{S}_\sigma$ for the remaining portion $\mathscr{S}_t$ of $\mathscr{S}_1$ and the right and left faces $\mathscr{A}_r$ and $\mathscr{A}_l$, of the rod, that is,

$$\mathscr{S}_d \cup \mathscr{S}_t = \mathscr{S}_1 \,,\; \mathscr{S}_d \cap \mathscr{S}_t = 0 \;;\; \mathscr{S}_\sigma = \mathscr{S} \cap \mathscr{S}_t = \mathscr{S}_d \cup \mathscr{A}_r \cup \mathscr{A}_l$$

In conjunction with Equations (2.5) and (3.1), we may write the deformation boundary conditions as

$$u_i^{(m,n)} - u_i^{*(m,n)} = 0 \,,\; \varphi_i^{(m,n)} - \varphi_i^{*(m,n)} = 0 \text{ on } \mathscr{S}_d \times [t_0, T) \tag{4.12}$$

Now, after multiplying by $x_2^m x_3^n$ Equations (2.6) are readily integrated, and then the traction boundary conditions are explicitly stated on the lateral surface:

$$S_k^{*(m,n)} - \nu_\alpha S_{\alpha k}^{(m,n)} = 0 \,,\; R_k^{*(m,n)} - \nu_\alpha R_{\alpha k}^{(m,n)} = 0 \;\text{ on }\; \mathscr{S}_t \times [t_0, T) \tag{4.13}$$

and on the faces $\mathscr{A}_r$ $(n_1 = +1)$ and $\mathscr{A}_l (n_1 = -1)$ of rod:

$$T_k^{*(m,n)} - n_1 T_{1k}^{(m,n)} = 0 \,,\; M_k^{*(m,n)} - n_1 M_{1k}^{(m,n)} = 0 \;\text{ on }\; \{\mathscr{A}_l, \mathscr{A}_r\} \times [t_0, T) \tag{4.14}$$

Here, ν_i is the unit exterior vector normal to the contour $\mathscr{C}$, and

$$\begin{aligned} \left(S_k^{*(m,n)}, R_k^{*(m,n)}\right) &= \oint_{\mathscr{C}} \nu_\alpha x_2^m x_3^n (t_{\alpha k}^*, m_{\alpha k}^*)\,ds \,, \\ \left(T_k^{*(m,n)}, M_k^{*(m,n)}\right) &= \int_{\mathscr{A}} x_2^m x_3^n (t_k^*, m_k^*)\,dA \end{aligned} \tag{4.15}$$

are also defined.

It is noteworthy, in conjunction with Equations (4.6) and (4.15), to have the relations:

$$S_k^{(m,n)} = \nu_\alpha S_{\alpha k}^{(m,n)}, \;\; R_k^{(m,n)} = \nu_\alpha R_{\alpha k}^{(m,n)} \tag{4.16}$$

4.4. *Initial conditions*

The foregoing equations are further supplemented by a set of initial conditions of the form:

$$\left.\begin{aligned} u_i^{(m,n)}(x_1, t_0) - \alpha_i^{*(m,n)}(x_1) = 0 \,,\; \dot{u}_i^{(m,n)}(x_1, t_0) - \beta_i^{*(m,n)}(x_1) = 0 \\ \varphi_i^{(m,n)}(x_1, t_0) - \gamma_i^{*(m,n)}(x_1) = 0 \,,\; \dot{\varphi}_i^{(m,n)}(x_1, t_0) - \delta_i^{*(m,n)}(x_1) = 0 \end{aligned}\right\} \text{ on } \mathscr{L}(t_0) \tag{4.17}$$

which follow readily from Equations (2.8) and (3.1).

4.5. *Theory of micropolar anisotropic rods*

The system of the kinematic relations (3.1–2), the macroscopic equations of motion (4.1), the constitutive relations (4.8), the boundary conditions, (4.12)–(4.14), and the initial conditions (4.17) constitutes the governing equations of a higher order, isothermal, linear theory of micropolar anisotropic rods. This system consists of an infinite number of equations in an infinite number of unknown functions; in order to construct a deterministic theory, we truncate the governing equations by the condition $m \in [0, M_0]$ and $n \in [0, N_0]$. Here, M_0 and N_0 are finite numbers which are chosen for a particular problem.

5. Example

To illustrate the possible truncation in the higher order theory of the previous section, we study here in detail the zeroth order isotropic theory and supplement this by a simple example. To begin with we explicitly state the linear isotropic theory of order (0, 0), that is to say, Bernoulli's theory of micropolar rods (cf., [2,5]). Following Dökmeci [2] for nonpolar case, we retain only $T_{11}^{(0,0)}$, $M_{11}^{(0,0)}$ and $\varepsilon_{(kk)}^{(0,0)}$, $\gamma_{(kk)}^{(0,0)}$, and set all the remaining rod quantities including temperatures equal to zero. Hence we may write out the constitutive relations (4.11) in the form:

$$\left.\begin{aligned} T_{11}^{(0,0)} &= A\left[\lambda\varepsilon_{kk}^{(0,0)} + (2\mu+\kappa)\varepsilon_{11}^{(0,0)}\right] = \mathscr{N} \\ T_{(\alpha\alpha)}^{(0,0)} &= A\left[\lambda\varepsilon_{kk}^{(0,0)} + (2\mu+\kappa)\varepsilon_{(\alpha\alpha)}^{(0,0)}\right] = 0 \\ M_{11}^{(0,0)} &= A\left[\alpha\gamma_{rr}^{(0,0)} + (\beta+\gamma)\gamma_{11}^{(0,0)}\right] = \mathscr{M} \\ M_{(\alpha\alpha)}^{(0,0)} &= A\left[\alpha\gamma_{rr}^{(0,0)} + (\beta+\gamma)\gamma_{(\alpha\alpha)}^{(0,0)}\right] = 0 \end{aligned}\right\} \tag{5.1}$$

the strain-deformation relationships (3.3):

$$\left.\begin{aligned} &\varepsilon_{11}^{(0,0)} = u_1'^{(0,0)}, \quad \varepsilon_{22}^{(0,0)} = u_2^{(1,0)}, \quad \varepsilon_{33}^{(0,0)} = u_3^{(0,1)} \\ &\gamma_{11}^{(0,0)} = \varphi_1'^{(0,0)}, \quad \gamma_{22}^{(0,0)} = \varphi_2^{(1,0)}, \quad \gamma_{33}^{(0,0)} = \varphi_3^{(0,1)} \end{aligned}\right\}. \tag{5.2}$$

and the rod equations of motion (4.1):

$$\mathscr{N}' + P - \rho A\ddot{w} = 0, \quad \mathscr{M}' + Q - \rho J\ddot{\phi} = 0 \tag{5.3}$$

where

$$w = u_1^{(0,0)}, \quad P = P_1^{(0,0)}, \quad \phi = \varphi_1^{(0,0)}, \quad Q = Q_1^{(0,0)}. \tag{5.4}$$

With the help of the second and fourth of Equations (5.1), we compute the relations:

$$\varepsilon_{\alpha\alpha}^{(0,0)} = -\frac{2\lambda\varepsilon_{11}^{(0,0)}}{(2\lambda+2\mu+\kappa)}, \quad \gamma_{\alpha\alpha}^{(0,0)} = -\frac{2\alpha\gamma_{11}^{(0,0)}}{2\alpha+\beta+\gamma}. \tag{5.5}$$

Inserting these relations into the first and third of Equations (5.1) and using Equations (5.2) and (5.4), we obtain $\mathscr{N}$ and $\mathscr{M}$ as follows:

$$\mathscr{N} = AE_0 w', \quad \mathscr{M} = AF_0\phi' \tag{5.6}$$

where

$$E_0 = \frac{\mu_0(2\mu_0+3\lambda)}{\mu_0+\lambda}, \quad F_0 = \frac{\beta_0(\beta_0+3\alpha)}{2\alpha+\beta_0} \tag{5.7a}$$

together with the relations [3]:

$$3\alpha+\beta_0 \geqq 0, \; \mu_0 \equiv \mu+\kappa/2 \geqq 0, \; 2\gamma \geqq \beta_0 \equiv \beta+\gamma \geqq 0, \; 2\mu_0+3\lambda \geqq 0, \; \kappa \geqq 0, \; \gamma \geqq 0 \tag{5.7b}$$

Thus, Equations (5.3) and (5.6) enable us to write Bernoulli's equations of motion for micropolar rods in terms of the deformation components, of the form:

$$\left.\begin{aligned} w'' &+ \frac{P}{AE_0} - \frac{1}{C_1^2}\ddot{w} = 0 \\ \phi'' &= \frac{Q}{AF_0} - \frac{1}{C_2^2}\ddot{\phi} = 0 \end{aligned}\right\} \tag{5.8}$$

with

$$C_1^2 = E_0/\rho\ ,\ C_2^2 = F_0/\rho \tag{5.9}$$

Here, with vanishing body forces and surface tractions, the first equation is the familiar one-dimensional wave equation of rods, and the second equation is the novelty of the present theory. Obviously, these equations are uncoupled; they become, however, coupled in the case of either the anisotropic solid or the higher order theories of micropolar rods.

For the harmonic waves with small amplitudes, propagating in the direction of the rod axis, $x_1 \equiv z$, we have

$$\{w, \phi\} = \{w_0, \phi_0\} \exp 2\pi i(qz+pt) \tag{5.10}$$

in which w_0 and ϕ_0 are constants, p is the frequency of the motion, q the wave number, $v = p/q$ the wave speed, and $\xi = 1/q$ the wave length. A substitution of Equations (5.10) into the one-dimensional wave equations yields a longitudinal displacement wave propagating with speed C_1, and a longitudinal microrotation wave propagating with speed C_2 whenever $2\alpha+\beta_0 > 0$, which reveals that all pulse shapes propagate without dispersion. For a material in which $\lambda = \mu + \kappa/2$ or $\nu = .25$, the wave speed C_1 reduces to

$$C_1 = v_0 = (E/\rho)^{\frac{1}{2}} = [2\mu(1+\nu)/\rho]^{\frac{1}{2}} \tag{5.11}$$

which is the usual rod velocity (e.g., [10]). Here, E is Young's modulus and ν Poisson's ratio.

6. Uniqueness

The standard device of establishing uniqueness in linear elasticity, that is, the consideration of the difference between two solutions arising from the same data, may be traced back to Fourier. The classical result for uniqueness in elasticity is credicted to Kirchhoff and its analogue in elastodynamics to Neumann; both of the results rely on the positive-definiteness of energy. In constructing uniqueness theorems, mention should also be made of logarithmic convexity arguments, methods involving reflection principle, and Holmgren's theorem. Among these, the classical energy argument, due to its familiarity and relative simplicity, is currently of wide use in the literature (see, for instance, [11] and the references cited therein). Nevertheless, without imposing a definiteness condition on the energy, we prefer the newer logarithmic convexity argument for its own intrinsic interest in the proof of the following theorem for the solutions of an initial-mixed boundary value problem. This problem is specified by the governing equations of the linear theory of micropolar anisotropic rods.

Theorem:

Given a regular region of finite rod space* $\mathscr{V}+\mathscr{S}$ *with boundary* $\mathscr{S}$ $(\mathscr{S}=\mathscr{S}_d \cup \mathscr{S}_\sigma, \mathscr{S}_d \cap \mathscr{S}_\sigma = 0)$ *in a Euclidean* 3*-space, then there exists at most one set of twice continuously differentiable vector functions* u_i *and* φ_i *in* $\mathscr{V}+\mathscr{S}$ *at* $t_0 \leqq t \leqq T$, *obeying Equations* (3.1–2), (4.1) *and* (4.8), *satisfying the boundary conditions* (4.12)–(4.14), *and the initial conditions* (4.17).

The proof utilizes the technique due to Knops and Payne [13] for uniqueness in 3-dimensional elastodynamics. As usual, we suppose the existence of two possible solutions d_i^1 and d_i^2 and set $d_i = d_i^1 - d_i^2$, that is, as before, $u_i = u_i^1 - u_i^2$ and $\varphi_i = \varphi_i^1 - \varphi_i^2$. By virtue of the linearity of the governing equations, this difference system clearly satisfies the homogeneous parts of the governing equations. We henceforth deal with the homogeneous equations corresponding to the difference system. In proving the theorem, it apparently suffices to show that the homogeneous problem may possess only the trivial solution.

Before proceeding further, we calculate the total energy of the rod. Let K and W denote the kinetic and potential energy densities, respectively. Thus, the kinetic and potential energies per unit length of the rod, V and U, may be expressed as

* The term regular is used in the sense of Kellogg [12].

$$V = \int_{\mathscr{A}} K\,dA\,, \quad U = \int_{\mathscr{A}} W\,dA \tag{6.1}$$

where

$$K = \tfrac{1}{2}\rho(\dot{u}_k \dot{u}_k + J_{kl}\dot{\phi}_k \dot{\phi}_l)\,, \quad W = \tfrac{1}{2}(t_{kl}\varepsilon_{kl} + m_{kl}\gamma_{lk}) \tag{6.2}$$

With the aid of Equations (3.1–3) and (4.2–4), these energies are found to be

$$\begin{aligned} V &= \sum_{m=0}^{M}\sum_{n=0}^{N} \tfrac{1}{2}\rho(\dot{U}_k^{(m,n)}\dot{u}_k^{(m,n)} + J_{kl}\dot{\phi}_k^{(m,n)}\dot{\varphi}_l^{(m,n)}) \\ U &= \sum_{m=0}^{M}\sum_{n=0}^{N} \tfrac{1}{2}[T_{1k}^{(m,n)}u_k'^{(m,n)} + (m\,T_{2k}^{(m-1,n)} + n\,T_{3k}^{(m,n-1)})u_k^{(m,n)} + \varepsilon_{lkp}T_{kl}^{(m,n)}\varphi_p^{(m,n)} \\ &\quad + M_{1k}^{(m,n)}\varphi_k'^{(m,n)} + (m\,M_{2k}^{(m-1,n)} + n\,M_{3k}^{(m,n-1)})\varphi_k^{(m,n)}] \end{aligned} \tag{6.3}$$

Hence, we have the total energy of the rod, Ω, is of the form:

$$\Omega = \int_0^L (V+U)\,dz \tag{6.4}$$

Also, it is worthwhile to note Schwartz's inequality in terms of functions $f_i(x_k, t)$ and $g_i(x_k, t)$, that is,

$$\int_{\mathscr{V}} f_k g_k\,dv \leqq \left\{\int_{\mathscr{V}} f_k f_k\,dv \int_{\mathscr{V}} g_k g_k\,dv\right\}^{\frac{1}{2}} \tag{6.5}$$

for later use.

Now, to establish uniqueness by logarithmic convexity arguments, we define the function $G(t)$ by

$$G(t) = \log H(t)\,, \quad t_0 \leqq t_1 < t < t_2 \leqq T \tag{6.6}$$

where

$$H(t) = \int_{\mathscr{V}} \tfrac{1}{2}\rho(u_k u_k + J_{kl}\varphi_k\varphi_l)\,dv\,, \qquad t_1 < t < t_2 \tag{6.7}$$

For all $t\in[t_0, t_1]$ and $t\in(t_2, T)$, without loss of generality, we may take $H(t)=0$ which clearly implies uniqueness. Hence we consider only an interval (t_1, t_2) of $[t_0, T)$, in which $H(t)>0$. Replacing the deformation components by Equations (3.1) and using Equation (4.4), we get

$$H = \int_0^L \tfrac{1}{2}\rho \sum_{m=0}^{M}\sum_{n=0}^{N} (U_k^{(m,n)}u_k^{(m,n)} + J_{kl}\phi_k^{(m,n)}\varphi_l^{(m,n)})\,dz \tag{6.8}$$

in this interval. A time differentiation of Equation (6.8) yields

$$\dot{H} = \int_0^L \rho \sum_{m=0}^{M}\sum_{n=0}^{N} (\dot{U}_k^{(m,n)}u_k^{(m,n)} + J_{kl}\dot{\phi}_k^{(m,n)}\varphi_l^{(m,n)})\,dz \tag{6.9}$$

and so for $\ddot{H}$

$$\ddot{H} = \int_0^L \left[2V + \rho\sum_{m=0}^{M}\sum_{n=0}^{N} (\ddot{U}_k^{(m,n)}u_k^{(m,n)} + J_{kl}\ddot{\phi}_k^{(m,n)}\varphi_l^{(m,n)})\right] dz \tag{6.10}$$

in which Equation (6.3) is used. In the equations above, the smoothness of the considered functions is tacitly assumed. In the last two terms on the right of Equation (6.10), we may substitute for the acceleration components from Equations (4.1) to obtain,

$$\begin{aligned} \ddot{H} = \int_0^L \Big\{ 2V + \sum_{m=0}^{M}\sum_{n=0}^{N} &[(T_{1k}'^{(m,n)} - m\,T_{2k}^{(m-1,n)} - n\,T_{3k}^{(m,n-1)} + S_k^{(m,n)})u_k^{(m,n)} \\ &+ (M_{1k}'^{(m,n)} - m\,M_{2k}^{(m-1,n)} - n\,M_{3k}^{(m,n-1)} + \varepsilon_{klp}T_{lp}^{(m,n)} + R_k^{(m,n)})\varphi_k^{(m,n)}]\Big\}\,dz\,. \end{aligned} \tag{6.11}$$

A comparison of this equation with Equation (6.3), followed by substitution from Equation (6.4), and an integration by parts leads to

$$\ddot{H} = -2\Omega + \int_0^L 4V\,dz + \varkappa + F \tag{6.12a}$$

with

$$\left.\begin{aligned} \varkappa &= \int_0^L \sum_{m=0}^{M} \sum_{n=0}^{N} \nu_\alpha (S_{\alpha k}^{(m,n)} u_k^{(m,n)} + R_{\alpha k}^{(m,n)} \varphi_k^{(m,n)}) dz \\ F &= \sum_{m=0}^{M} \sum_{n=0}^{N} (T_{1k}^{(m,n)} u_k^{(m,n)} + M_{1k}^{(m,n)} \varphi_k^{(m,n)}) \Big|_{z=0}^{L} \end{aligned}\right\} \tag{6.12b}$$

where Equations (4.16) are used. By virtue of conservation of energy and the initial conditions (4.17), the total energy of the system is equal to zero, i.e., $\Omega = 0$. Further, we consider a case where F and $\varkappa$ vanish. Thus, $\ddot{H}$ finally becomes:

$$\ddot{H} = 2 \int_0^L \sum_{m=0}^{M} \sum_{n=0}^{N} \rho (\dot{U}_k^{(m,n)} \dot{u}_k^{(m,n)} + J_{kl} \dot{\phi}_k^{(m,n)} \dot{\varphi}_l^{(m,n)}) dz \tag{6.13}$$

in which the first of Equations (6.3) is used.

At this point, using the derived results we form the relation:

$$\begin{aligned} H^2 \ddot{G} = H\ddot{H} - \dot{H}^2 = &\left\{ \int_0^L \sum_{m=0}^{M} \sum_{n=0}^{N} \rho (\dot{U}_k^{(m,n)} \dot{u}_k^{(m,n)} + J_{kl} \dot{\phi}_k^{(m,n)} \dot{\varphi}_l^{(m,n)}) dz \right\} \cdot \\ &\cdot \left\{ \int_0^L \sum_{m=0}^{M} \sum_{n=0}^{N} \rho (U_k^{(m,n)} u_k^{(m,n)} + J_{kl} \phi_k^{(m,n)} \varphi_l^{(m,n)}) dz \right\} \\ &- \left\{ \int_0^L \sum_{m=0}^{M} \sum_{n=0}^{N} \rho (\dot{U}_k^{(m,n)} u_k^{(m,n)} + J_{kl} \dot{\phi}_k^{(m,n)} \varphi_l^{(m,n)}) dz \right\}^2 \end{aligned} \tag{6.14}$$

for the logarithmic convexity of G. In view of Schwartz's inequality (6.5) and Equation (4.4), we may conclude that

$$\ddot{G} \equiv \frac{H\ddot{H} - \dot{H}^2}{H^2} \equiv \left(\frac{\dot{H}}{H}\right)^{\cdot} \leqq 0 \tag{6.15}$$

on the interval (t_1, t_2). It may be shown, after some manipulation, that this results in

$$H(t) \leqq [H(t_1)]^{(t_2 - t)/(t_2 - t_1)} [H(t_2)]^{(t - t_1)/(t_2 - t_1)} \tag{6.16}$$

Since by continuity $H(t_1) = 0$, Equation (6.16) immediately shows that $H(t) = 0$ for $t_1 \leqq t \leqq t_2$, and that $H(t) = 0$ for all $t \in [t_0, T)$. Therefore, the continuity of u_i and φ_i on $\mathscr{V}$ implies that they both vanish identically on $\mathscr{V}$, and thus proving the theorem for the aforementioned case, i.e., $\varkappa = F = 0$.

Lastly, attention is focused to the proper conditions which render $\varkappa$ and F to zero. The terms of $\varkappa$ and F consist of the inner products of traction- and deformation difference components on the lateral boundary and the faces, respectively. To begin with, it is evident that the boundary conditions (4.12–14) make both $\varkappa$ and F zero. Further, to specify one member of each of the products:

$$T_{1k}^{(m,n)} u_k^{(m,n)}, \quad M_{1k}^{(m,n)} \varphi_k^{(m,n)} \tag{6.17a}$$

at each end $x_1 = 0$ and $x_1 = L$, and one member of each of the products:

$$\nu_\alpha S_{\alpha k}^{(m,n)} u_k^{(m,n)}, \quad \nu_\alpha R_{\alpha k}^{(m,n)} \varphi_k^{(m,n)} \tag{6.17b}$$

at each point z along the length of rod for each pair of $m \in [0, M]$ and $n \in [0, N]$ in any solution starting from the initial conditions (4.17), (mixed-mixed boundary conditions) are sufficient to ensure a unique solution for the initial mixed boundary value problem in question.

7. Conclusions

In the preceding work which was motivated by the kinematics appropriate to a micropolar rod, together with a suitable averaging procedure, we have established a dynamic, higher order and deterministic theory for isothermal anisotropic rods. The one-dimensional, approximate governing equations of the theory have been formulated by a consistent and systematic reduction of the 3-D micropolar elastodynamics. By the proper truncation of the series, that is, selecting M and N, for particular applications, these equations incorporate as many higher order effects as deemed necessary. Hence the usual correction factors of rods (see, e.g., [7]) have been abrogated in a rational way. The theory presented governs all the types of motion of micropolar as well as non-polar rods of uniform cross-sections.

It is worth remarking that the derivation of the kinematics and dynamical balance laws was independent of the rod material. The constitutive equations of Section 4 have been developed for linear micropolar elastic materials. However, the treatment of other classes of materials follows readily. In particular, for linear viscoelastic materials the elastic moduli in Equations (4.8) and (4.11) should be replaced by their corresponding Stieltjes convolutions given in [14], (see, e.g., [10] for non-polar case). Further, one needs simply to follow [2] in constructing the non-linear theory.

Bernoulli's theory of micropolar rods, as a special case of the isotropic theory, has been given explicitly. By the use of this theory, then the longitudinal propagation of waves has been studied in detail. The arising of new class of waves due to microrotational motions has been demonstrated. This class of waves has its own speeds. Though they were uncoupled in our example, they become, in general, coupled and dispersive.

Moreover, the uniqueness in the deterministic theory has been examined. A dynamic uniqueness theorem has been proved for the solutions of the initial mixed-boundary value problem defined by the governing equations. Our method of proof was based upon logarithmic convexity arguments (cf. [15] for piezoelectric crystal bars). Hence the use of definiteness assumption for the strain energy density has been excluded here. Apparently, this technique may be adopted to similar treatments of other one- and two-dimensional approximate continuum theories.

Furthermore, it may be noted that all of the results presented are in full agreement with the linear theory of non-polar rods [2,16] if the terms involving with couple stresses are dropped out. Finally, as stated earlier, the thermodynamical considerations are left untreated (see, e.g., [17] for a discussion of 2-D continuum theory). A very general treatment, including irreversible stresses, is left for a future study.

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